Dynamic Semantics and Discourse

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Introduction

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An old problem:

A man enters the room. He smiles.

$[\![$A man enters the room$]\!] = \exists x.\mathbf{man}(x) \wedge \mathbf{enters_the_room}(x)$. x is bound.

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A well known solution: DRT.

- The reference markers of DRT act as existential quantifiers.
- Nevertheless, from a technical point of view, they must be considered as free variables.

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- We will interpret a sentence according to both its left and right contexts.
- These two kinds of contexts will be abstracted over the meaning of the sentences.

Typing the left and the right contexts

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Montague semantics is based on Church's simple type theory, which provides a full hierarchy of functional types built upon two atomic types:

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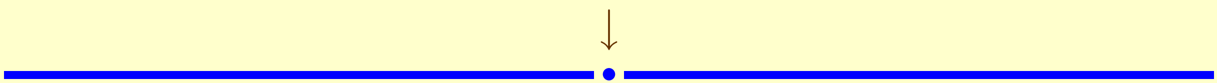

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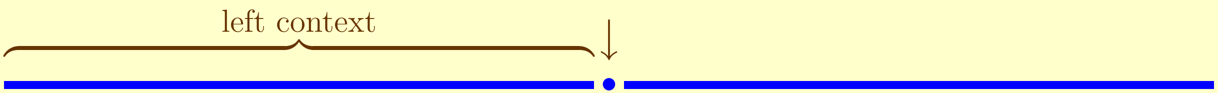

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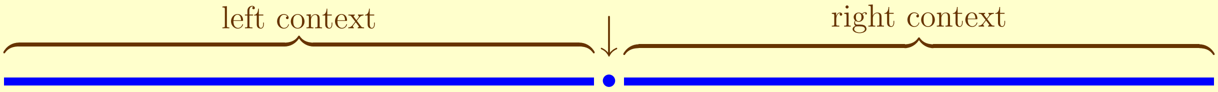

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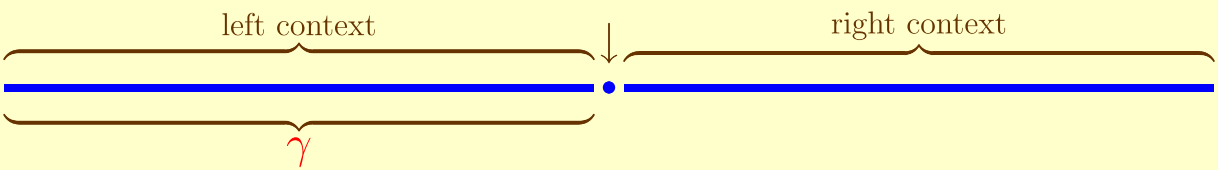

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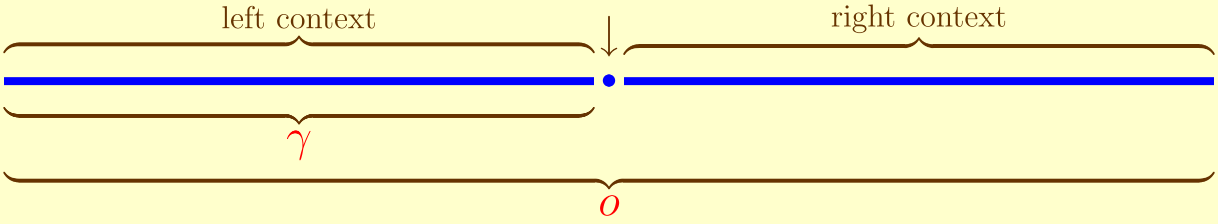

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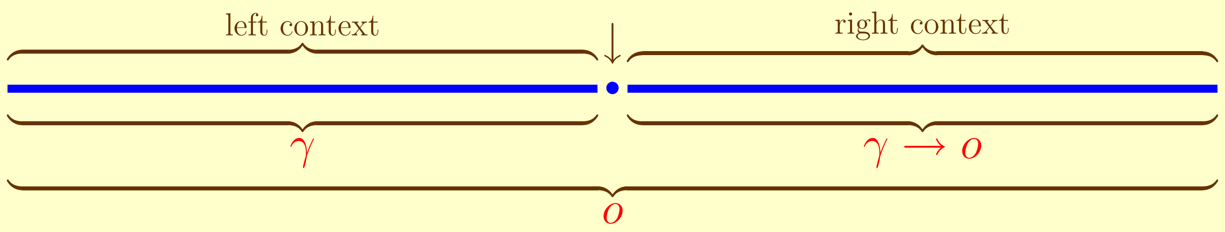

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$$[\![S_1 . S_2]\!] = \lambda e \phi . [\![S_1]\!] \, e \, (\lambda e' . [\![S_2]\!] \, e' \, \phi)$$

Note that this operation is associative!

Back to DRT and DRSs

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Consider a DRS:

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To such a structure, corresponds the following λ-term of type $\gamma \rightarrow (\gamma \rightarrow o) \rightarrow o$:

$$\lambda e \phi . \exists x_1 \ldots x_n . C_1 \wedge \cdots \wedge C_m \wedge \phi \, e'$$

where e' is a context made of e and of the variables $x_1, \ldots, x_n$.

Updating and accessing the context

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$\lambda e\phi.\,[\![\text{John loves Mary}]\!]\,e\,(\lambda e'.\,[\![\text{He smiles at her}]\!]\,e'\,\phi)$

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Assigning a semantics to the lexical entries

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$$
\begin{aligned}
[\![s]\!] &= o \\
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$$\begin{aligned} [\![s]\!] &= o \\ [\![n]\!] &= \iota \rightarrow o \\ [\![np]\!] &= (\iota \rightarrow o) \rightarrow o \end{aligned}$$

$$[\![s]\!] = o \quad (1)$$

$$[\![n]\!] = \iota \rightarrow [\![s]\!] \quad (2)$$

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Replacing (1) with:

$$[\![s]\!] = \gamma \rightarrow (\gamma \rightarrow o) \rightarrow o$$

we obtain:

$$\begin{aligned} [\![n]\!] &= \iota \rightarrow \gamma \rightarrow (\gamma \rightarrow o) \rightarrow o \\ [\![np]\!] &= (\iota \rightarrow \gamma \rightarrow (\gamma \rightarrow o) \rightarrow o) \rightarrow \gamma \rightarrow (\gamma \rightarrow o) \rightarrow o \end{aligned}$$

Nouns

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$$[\![n]\!] = \iota \rightarrow \gamma \rightarrow (\gamma \rightarrow o) \rightarrow o$$

Nouns

$$[\![n]\!] = \iota \to \gamma \to (\gamma \to o) \to o$$

$$[\![\text{man}]\!] = \lambda x e \phi .\, \mathbf{man}\, x \wedge \phi\, e$$
$$[\![\text{woman}]\!] = \lambda x e \phi .\, \mathbf{woman}\, x \wedge \phi\, e$$
$$[\![\text{farmer}]\!] = \lambda x e \phi .\, \mathbf{farmer}\, x \wedge \phi\, e$$
$$[\![\text{donkey}]\!] = \lambda x e \phi .\, \mathbf{donkey}\, x \wedge \phi\, e$$

Noun phrases

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$$[\![np]\!] = (\iota \to \gamma \to (\gamma \to o) \to o) \to \gamma \to (\gamma \to o) \to o$$

Noun phrases

$$[\![np]\!] = (\iota \to \gamma \to (\gamma \to o) \to o) \to \gamma \to (\gamma \to o) \to o$$

$$[\![\text{John}]\!] = \lambda \psi e \phi .\, \psi\, \mathbf{j}\, e\, (\lambda e.\, \phi\, (\mathbf{j} {::} e))$$

$$[\![\text{Mary}]\!] = \lambda \psi e \phi .\, \psi\, \mathbf{m}\, e\, (\lambda e.\, \phi\, (\mathbf{m} {::} e))$$

$$[\![\text{he}]\!] = \lambda \psi e \phi .\, \psi\, (\mathtt{sel}_{he}\, e)\, e\, \phi$$

$$[\![\text{her}]\!] = \lambda \psi e \phi .\, \psi\, (\mathtt{sel}_{her}\, e)\, e\, \phi$$

$$[\![\text{it}]\!] = \lambda \psi e \phi .\, \psi\, (\mathtt{sel}_{it}\, e)\, e\, \phi$$

Determiners

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$$[\![det]\!] = [\![n]\!] \longrightarrow [\![np]\!]$$

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$$[\![det]\!] = [\![n]\!] \longrightarrow [\![np]\!]$$

$[\![\text{a}]\!] = \lambda n \psi e \phi .\, \exists x .\, n\, x\, e\, (\lambda e .\, \psi\, x\, (x{::}e)\, \phi)$

$[\![\text{every}]\!] = \lambda n \psi e \phi .\, (\forall x .\, \neg(n\, x\, e\, (\lambda e .\, \neg(\psi\, x\, (x{::}e)\, (\lambda e .\, \top))))) \, \wedge \, \phi\, e$

Transitive verbs

Transitive verbs

$$[\![tv]\!] = [\![np]\!] \longrightarrow [\![np]\!] \longrightarrow [\![s]\!]$$

Transitive verbs

$$[\![tv]\!] = [\![np]\!] \rightarrow [\![np]\!] \rightarrow [\![s]\!]$$

$$\begin{aligned}
[\![\text{loves}]\!] &= \lambda os.\, s\,(\lambda x.\, o\,(\lambda ye\phi.\, \mathbf{love}\, x\, y \;\wedge\; \phi\, e)) \\
[\![\text{owns}]\!] &= \lambda os.\, s\,(\lambda x.\, o\,(\lambda ye\phi.\, \mathbf{own}\, x\, y \;\wedge\; \phi\, e)) \\
[\![\text{beats}]\!] &= \lambda os.\, s\,(\lambda x.\, o\,(\lambda ye\phi.\, \mathbf{beat}\, x\, y \;\wedge\; \phi\, e))
\end{aligned}$$

Relative pronouns

Relative pronouns

$$[\![rel]\!] = ([\![np]\!] \longrightarrow [\![s]\!]) \longrightarrow [\![n]\!] \longrightarrow [\![n]\!]$$

Relative pronouns

$$[\![rel]\!] = ([\![np]\!] \longrightarrow [\![s]\!]) \longrightarrow [\![n]\!] \longrightarrow [\![n]\!]$$

$$[\![\text{who}]\!] = \lambda rnxe\phi.\, n\, x\, e\, (\lambda e.\, r\, (\lambda \psi.\, \psi\, x)\, e\, \phi)$$

$[\![\text{beats}]\!]\ [\![\text{it}]\!]\ ([\![\text{every}]\!]\ ([\![\text{who}]\!]\ ([\![\text{owns}]\!]\ ([\![\text{a}]\!]\ [\![\text{donkey}]\!]))\ [\![\text{farmer}]\!]))$

$[\![\text{beats}]\!]\ [\![\text{it}]\!]\ ([\![\text{every}]\!]\ ([\![\text{who}]\!]\ ([\![\text{owns}]\!]\ ([\![\text{a}]\!]\ [\![\text{donkey}]\!]))\ [\![\text{farmer}]\!]))$

$[\![\text{a}]\!]\ [\![\text{donkey}]\!]$

$[\![\text{beats}]\!]\ [\![\text{it}]\!]\ ([\![\text{every}]\!]\ ([\![\text{who}]\!]\ ([\![\text{owns}]\!]\ ([\![\text{a}]\!]\ [\![\text{donkey}]\!]))\ [\![\text{farmer}]\!]))$

$$
\begin{aligned}
&[\![\text{a}]\!]\ [\![\text{donkey}]\!] \\
&\quad = \quad (\lambda n\psi e\phi.\, \exists y.\, n\, y\, e\, (\lambda e.\, \psi\, y\, (y{::}e)\, \phi))\ [\![\text{donkey}]\!]
\end{aligned}
$$

$[\![\text{beats}]\!]\,[\![\text{it}]\!]\,([\![\text{every}]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))$

$$
\begin{aligned}
&[\![\text{a}]\!]\,[\![\text{donkey}]\!] \\
&\quad = \quad (\lambda n\psi e\phi.\,\exists y.\,n\,y\,e\,(\lambda e.\,\psi\,y\,(y{::}e)\,\phi))\,[\![\text{donkey}]\!] \\
&\quad \twoheadrightarrow_\beta \lambda\psi e\phi.\,\exists y.\,[\![\text{donkey}]\!]\,y\,e\,(\lambda e.\,\psi\,y\,(y{::}e)\,\phi)
\end{aligned}
$$

$[\![\text{beats}]\!]\,[\![\text{it}]\!]\,([\![\text{every}]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))$

$$
\begin{aligned}
&[\![\text{a}]\!]\,[\![\text{donkey}]\!] \\
&\quad = \quad (\lambda n\psi e\phi.\, \exists y.\, n\, y\, e\, (\lambda e.\, \psi\, y\, (y{::}e)\, \phi))\, [\![\text{donkey}]\!] \\
&\quad \twoheadrightarrow_\beta \lambda \psi e\phi.\, \exists y.\, [\![\text{donkey}]\!]\, y\, e\, (\lambda e.\, \psi\, y\, (y{::}e)\, \phi) \\
&\quad = \quad \lambda \psi e\phi.\, \exists y.\, (\lambda x e\phi.\, \mathbf{donkey}\, x \wedge \phi\, e)\, y\, e\, (\lambda e.\, \psi\, y\, (y{::}e)\, \phi)
\end{aligned}
$$

$[\![\text{beats}]\!]\,[\![\text{it}]\!]\,([\![\text{every}]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))$

$$
\begin{aligned}
&[\![\text{a}]\!]\,[\![\text{donkey}]\!] \\
&\quad = \;(\lambda n\psi e\phi.\,\exists y.\,n\,y\,e\,(\lambda e.\,\psi\,y\,(y{::}e)\,\phi))\,[\![\text{donkey}]\!] \\
&\quad \twoheadrightarrow_\beta \lambda\psi e\phi.\,\exists y.\,[\![\text{donkey}]\!]\,y\,e\,(\lambda e.\,\psi\,y\,(y{::}e)\,\phi) \\
&\quad = \;\lambda\psi e\phi.\,\exists y.\,(\lambda x e\phi.\,\mathbf{donkey}\,x \wedge \phi\,e)\,y\,e\,(\lambda e.\,\psi\,y\,(y{::}e)\,\phi) \\
&\quad \twoheadrightarrow_\beta \lambda\psi e\phi.\,\exists y.\,\mathbf{donkey}\,y \wedge (\lambda e.\,\psi\,y\,(y{::}e)\,\phi)\,e
\end{aligned}
$$

$[\![\text{beats}]\!]\,[\![\text{it}]\!]\,([\![\text{every}]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))$

$$
\begin{aligned}
&[\![\text{a}]\!]\,[\![\text{donkey}]\!] \\
&\quad = \; (\lambda n\psi e\phi.\, \exists y.\, n\, y\, e\, (\lambda e.\, \psi\, y\, (y{::}e)\, \phi))\, [\![\text{donkey}]\!] \\
&\quad \twoheadrightarrow_\beta \lambda\psi e\phi.\, \exists y.\, [\![\text{donkey}]\!]\, y\, e\, (\lambda e.\, \psi\, y\, (y{::}e)\, \phi) \\
&\quad = \; \lambda\psi e\phi.\, \exists y.\, (\lambda x e\phi.\, \mathbf{donkey}\, x \wedge \phi\, e)\, y\, e\, (\lambda e.\, \psi\, y\, (y{::}e)\, \phi) \\
&\quad \twoheadrightarrow_\beta \lambda\psi e\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge (\lambda e.\, \psi\, y\, (y{::}e)\, \phi)\, e \\
&\quad \twoheadrightarrow_\beta \lambda\psi e\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \psi\, y\, (y{::}e)\, \phi
\end{aligned}
$$

$[\![\text{beats}]\!]\,[\![\text{it}]\!]\,([\![\text{every}]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))$

$[\![\text{a}]\!]\,[\![\text{donkey}]\!]$

$$\begin{aligned}
&= (\lambda n\psi e\phi.\, \exists y.\, n\, y\, e\, (\lambda e.\, \psi\, y\, (y{::}e)\, \phi))\, [\![\text{donkey}]\!] \\
&\twoheadrightarrow_\beta \lambda\psi e\phi.\, \exists y.\, [\![\text{donkey}]\!]\, y\, e\, (\lambda e.\, \psi\, y\, (y{::}e)\, \phi) \\
&= \lambda\psi e\phi.\, \exists y.\, (\lambda x e\phi.\, \mathbf{donkey}\, x \wedge \phi\, e)\, y\, e\, (\lambda e.\, \psi\, y\, (y{::}e)\, \phi) \\
&\twoheadrightarrow_\beta \lambda\psi e\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge (\lambda e.\, \psi\, y\, (y{::}e)\, \phi)\, e \\
&\twoheadrightarrow_\beta \lambda\psi e\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \psi\, y\, (y{::}e)\, \phi
\end{aligned}$$

$[\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!])$

$[\![\text{beats}]\!]\ [\![\text{it}]\!]\ ([\![\text{every}]\!]\ ([\![\text{who}]\!]\ ([\![\text{owns}]\!]\ ([\![\text{a}]\!]\ [\![\text{donkey}]\!]))\ [\![\text{farmer}]\!]))$

$[\![\text{a}]\!]\ [\![\text{donkey}]\!]$

$$\begin{aligned}
&= \ (\lambda n\psi e\phi.\, \exists y.\, n\, y\, e\, (\lambda e.\, \psi\, y\, (y{::}e)\, \phi))\ [\![\text{donkey}]\!] \\
&\twoheadrightarrow_\beta \lambda\psi e\phi.\, \exists y.\, [\![\text{donkey}]\!]\, y\, e\, (\lambda e.\, \psi\, y\, (y{::}e)\, \phi) \\
&= \ \lambda\psi e\phi.\, \exists y.\, (\lambda x e\phi.\, \mathbf{donkey}\, x \wedge \phi\, e)\, y\, e\, (\lambda e.\, \psi\, y\, (y{::}e)\, \phi) \\
&\twoheadrightarrow_\beta \lambda\psi e\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge (\lambda e.\, \psi\, y\, (y{::}e)\, \phi)\, e \\
&\twoheadrightarrow_\beta \lambda\psi e\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \psi\, y\, (y{::}e)\, \phi
\end{aligned}$$

$[\![\text{owns}]\!]\ ([\![\text{a}]\!]\ [\![\text{donkey}]\!])$

$$= [\![\text{owns}]\!]\ (\lambda\psi e\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \psi\, y\, (y{::}e)\, \phi)$$

$[\![\text{beats}]\!]\, [\![\text{it}]\!]\, ([\![\text{every}]\!]\, ([\![\text{who}]\!]\, ([\![\text{owns}]\!]\, ([\![\text{a}]\!]\, [\![\text{donkey}]\!]))\, [\![\text{farmer}]\!]))$

$[\![\text{a}]\!]\, [\![\text{donkey}]\!]$

$$\begin{aligned}
&= \quad (\lambda n\psi e\phi.\, \exists y.\, n\, y\, e\, (\lambda e.\, \psi\, y\, (y{::}e)\, \phi))\, [\![\text{donkey}]\!] \\
&\twoheadrightarrow_\beta \lambda\psi e\phi.\, \exists y.\, [\![\text{donkey}]\!]\, y\, e\, (\lambda e.\, \psi\, y\, (y{::}e)\, \phi) \\
&= \quad \lambda\psi e\phi.\, \exists y.\, (\lambda x e\phi.\, \mathbf{donkey}\, x \wedge \phi\, e)\, y\, e\, (\lambda e.\, \psi\, y\, (y{::}e)\, \phi) \\
&\twoheadrightarrow_\beta \lambda\psi e\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge (\lambda e.\, \psi\, y\, (y{::}e)\, \phi)\, e \\
&\twoheadrightarrow_\beta \lambda\psi e\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \psi\, y\, (y{::}e)\, \phi
\end{aligned}$$

$[\![\text{owns}]\!]\, ([\![\text{a}]\!]\, [\![\text{donkey}]\!])$

$$\begin{aligned}
&= [\![\text{owns}]\!]\, (\lambda\psi e\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \psi\, y\, (y{::}e)\, \phi) \\
&= (\lambda o s.\, s\, (\lambda x.\, o\, (\lambda y e\phi.\, \mathbf{own}\, x\, y \wedge \phi\, e)))\, (\lambda\psi e\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \psi\, y\, (y{::}e)\, \phi)
\end{aligned}$$

$[\![\text{beats}]\!]\,[\![\text{it}]\!]\,([\![\text{every}]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))$

$[\![\text{a}]\!]\,[\![\text{donkey}]\!]$

$$\begin{aligned}
&= (\lambda n\psi e\phi.\, \exists y.\, n\, y\, e\, (\lambda e.\, \psi\, y\, (y{::}e)\, \phi))\, [\![\text{donkey}]\!] \\
&\twoheadrightarrow_\beta \lambda\psi e\phi.\, \exists y.\, [\![\text{donkey}]\!]\, y\, e\, (\lambda e.\, \psi\, y\, (y{::}e)\, \phi) \\
&= \lambda\psi e\phi.\, \exists y.\, (\lambda x e\phi.\, \mathbf{donkey}\, x \wedge \phi\, e)\, y\, e\, (\lambda e.\, \psi\, y\, (y{::}e)\, \phi) \\
&\twoheadrightarrow_\beta \lambda\psi e\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge (\lambda e.\, \psi\, y\, (y{::}e)\, \phi)\, e \\
&\twoheadrightarrow_\beta \lambda\psi e\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \psi\, y\, (y{::}e)\, \phi
\end{aligned}$$

$[\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!])$

$$\begin{aligned}
&= [\![\text{owns}]\!]\, (\lambda\psi e\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \psi\, y\, (y{::}e)\, \phi) \\
&= (\lambda o s.\, s\, (\lambda x.\, o\, (\lambda y e\phi.\, \mathbf{own}\, x\, y \wedge \phi\, e)))\, (\lambda\psi e\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \psi\, y\, (y{::}e)\, \phi) \\
&\twoheadrightarrow_\beta \lambda s.\, s\, (\lambda x.\, (\lambda\psi e\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \psi\, y\, (y{::}e)\, \phi)\, (\lambda y e\phi.\, \mathbf{own}\, x\, y \wedge \phi\, e))
\end{aligned}$$

$[\![\text{beats}]\!]\,[\![\text{it}]\!]\,([\![\text{every}]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))$

$[\![\text{a}]\!]\,[\![\text{donkey}]\!]$

$$\begin{aligned}
&= \ (\lambda n\psi e\phi.\,\exists y.\,n\,y\,e\,(\lambda e.\,\psi\,y\,(y{::}e)\,\phi))\,[\![\text{donkey}]\!]\\
&\twoheadrightarrow_\beta \lambda\psi e\phi.\,\exists y.\,[\![\text{donkey}]\!]\,y\,e\,(\lambda e.\,\psi\,y\,(y{::}e)\,\phi)\\
&= \ \lambda\psi e\phi.\,\exists y.\,(\lambda x e\phi.\,\mathbf{donkey}\,x \wedge \phi\,e)\,y\,e\,(\lambda e.\,\psi\,y\,(y{::}e)\,\phi)\\
&\twoheadrightarrow_\beta \lambda\psi e\phi.\,\exists y.\,\mathbf{donkey}\,y \wedge (\lambda e.\,\psi\,y\,(y{::}e)\,\phi)\,e\\
&\twoheadrightarrow_\beta \lambda\psi e\phi.\,\exists y.\,\mathbf{donkey}\,y \wedge \psi\,y\,(y{::}e)\,\phi
\end{aligned}$$

$[\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!])$

$$\begin{aligned}
&= [\![\text{owns}]\!]\,(\lambda\psi e\phi.\,\exists y.\,\mathbf{donkey}\,y \wedge \psi\,y\,(y{::}e)\,\phi)\\
&= (\lambda os.\,s\,(\lambda x.\,o\,(\lambda ye\phi.\,\mathbf{own}\,x\,y \wedge \phi\,e)))\,(\lambda\psi e\phi.\,\exists y.\,\mathbf{donkey}\,y \wedge \psi\,y\,(y{::}e)\,\phi)\\
&\twoheadrightarrow_\beta \lambda s.\,s\,(\lambda x.\,(\lambda\psi e\phi.\,\exists y.\,\mathbf{donkey}\,y \wedge \psi\,y\,(y{::}e)\,\phi)\,(\lambda ye\phi.\,\mathbf{own}\,x\,y \wedge \phi\,e))\\
&\twoheadrightarrow_\beta \lambda s.\,s\,(\lambda xe\phi.\,\exists y.\,\mathbf{donkey}\,y \wedge (\lambda ye\phi.\,\mathbf{own}\,x\,y \wedge \phi\,e)\,y\,(y{::}e)\,\phi)
\end{aligned}$$

$[\![\text{beats}]\!]\, [\![\text{it}]\!]\, ([\![\text{every}]\!]\, ([\![\text{who}]\!]\, ([\![\text{owns}]\!]\, ([\![\text{a}]\!]\, [\![\text{donkey}]\!]))\, [\![\text{farmer}]\!]))$

$$
\begin{aligned}
&[\![\text{a}]\!]\, [\![\text{donkey}]\!] \\
&\quad = \ (\lambda n \psi e \phi.\, \exists y.\, n\, y\, e\, (\lambda e.\, \psi\, y\, (y{::}e)\, \phi))\, [\![\text{donkey}]\!] \\
&\quad \twoheadrightarrow_\beta \lambda \psi e \phi.\, \exists y.\, [\![\text{donkey}]\!]\, y\, e\, (\lambda e.\, \psi\, y\, (y{::}e)\, \phi) \\
&\quad = \ \lambda \psi e \phi.\, \exists y.\, (\lambda x e \phi.\, \mathbf{donkey}\, x \wedge \phi\, e)\, y\, e\, (\lambda e.\, \psi\, y\, (y{::}e)\, \phi) \\
&\quad \twoheadrightarrow_\beta \lambda \psi e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge (\lambda e.\, \psi\, y\, (y{::}e)\, \phi)\, e \\
&\quad \twoheadrightarrow_\beta \lambda \psi e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \psi\, y\, (y{::}e)\, \phi
\end{aligned}
$$

$$
\begin{aligned}
&[\![\text{owns}]\!]\, ([\![\text{a}]\!]\, [\![\text{donkey}]\!]) \\
&\quad = [\![\text{owns}]\!]\, (\lambda \psi e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \psi\, y\, (y{::}e)\, \phi) \\
&\quad = (\lambda o s.\, s\, (\lambda x.\, o\, (\lambda y e \phi.\, \mathbf{own}\, x\, y \wedge \phi\, e)))\, (\lambda \psi e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \psi\, y\, (y{::}e)\, \phi) \\
&\quad \twoheadrightarrow_\beta \lambda s.\, s\, (\lambda x.\, (\lambda \psi e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \psi\, y\, (y{::}e)\, \phi)\, (\lambda y e \phi.\, \mathbf{own}\, x\, y \wedge \phi\, e)) \\
&\quad \twoheadrightarrow_\beta \lambda s.\, s\, (\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge (\lambda y e \phi.\, \mathbf{own}\, x\, y \wedge \phi\, e)\, y\, (y{::}e)\, \phi) \\
&\quad \twoheadrightarrow_\beta \lambda s.\, s\, (\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\, (y{::}e))
\end{aligned}
$$

$[\![\text{who}]\!]\ ([\![\text{owns}]\!]\ ([\![\text{a}]\!]\ [\![\text{donkey}]\!]))$

$$
\begin{aligned}
&[\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))\\
&\quad = \quad [\![\text{who}]\!]\,(\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e)))
\end{aligned}
$$

$$
\begin{aligned}
&[\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))\\
&\quad = \quad [\![\text{who}]\!]\,(\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \,\wedge\, \mathbf{own}\, x\, y \,\wedge\, \phi\,(y{::}e)))\\
&\quad = \quad (\lambda r n x e \phi.\, n\, x\, e\,(\lambda e.\, r\,(\lambda \psi.\, \psi\, x)\, e\, \phi))\\
&\qquad\qquad\quad (\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \,\wedge\, \mathbf{own}\, x\, y \,\wedge\, \phi\,(y{::}e)))
\end{aligned}
$$

$$
\begin{aligned}
&[\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))\\
&\quad = \quad [\![\text{who}]\!]\,(\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e)))\\
&\quad = \quad (\lambda r n x e \phi.\, n\, x\, e\,(\lambda e.\, r\,(\lambda \psi.\, \psi\, x)\, e\, \phi))\\
&\qquad\qquad (\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e)))\\
&\quad \twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\\
&\qquad (\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e)))\,(\lambda \psi.\, \psi\, x)\, e\, \phi)
\end{aligned}
$$

$$[\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))$$

$$= \quad [\![\text{who}]\!]\,(\lambda s.\, s\,(\lambda xe\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e)))$$

$$= \quad (\lambda rnxe\phi.\, n\, x\, e\,(\lambda e.\, r\,(\lambda\psi.\, \psi\, x)\, e\, \phi))$$
$$(\lambda s.\, s\,(\lambda xe\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e)))$$

$$\twoheadrightarrow_\beta \lambda nxe\phi.\, n\, x\, e\,(\lambda e.$$
$$(\lambda s.\, s\,(\lambda xe\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e)))\,(\lambda\psi.\, \psi\, x)\, e\, \phi)$$

$$\twoheadrightarrow_\beta \lambda nxe\phi.\, n\, x\, e\,(\lambda e.\,(\lambda\psi.\, \psi\, x)\,(\lambda xe\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e))\, e\, \phi)$$

$$[\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))$$

$$\begin{aligned}
&= \quad [\![\text{who}]\!]\,(\lambda s.\, s\,(\lambda xe\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e)))\\
&= \quad (\lambda rnxe\phi.\, n\, x\, e\,(\lambda e.\, r\,(\lambda\psi.\, \psi\, x)\, e\, \phi))\\
&\qquad\quad (\lambda s.\, s\,(\lambda xe\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e)))\\
&\twoheadrightarrow_\beta \lambda nxe\phi.\, n\, x\, e\,(\lambda e.\\
&\qquad (\lambda s.\, s\,(\lambda xe\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e)))\,(\lambda\psi.\, \psi\, x)\, e\, \phi)\\
&\twoheadrightarrow_\beta \lambda nxe\phi.\, n\, x\, e\,(\lambda e.\, (\lambda\psi.\, \psi\, x)\,(\lambda xe\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e))\, e\, \phi)\\
&\twoheadrightarrow_\beta \lambda nxe\phi.\, n\, x\, e\,(\lambda e.\, (\lambda xe\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e))\, x\, e\, \phi)
\end{aligned}$$

$$
\begin{aligned}
&[\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))\\
&\quad= \quad [\![\text{who}]\!]\,(\lambda s.\, s\,(\lambda x e\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e)))\\
&\quad= \quad (\lambda r n x e \phi.\, n\, x\, e\, (\lambda e.\, r\, (\lambda \psi.\, \psi\, x)\, e\, \phi))\\
&\qquad\qquad (\lambda s.\, s\,(\lambda x e\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e)))\\
&\quad\twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\, (\lambda e.\\
&\qquad (\lambda s.\, s\,(\lambda x e\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e)))\,(\lambda \psi.\, \psi\, x)\, e\, \phi)\\
&\quad\twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\, (\lambda e.\, (\lambda \psi.\, \psi\, x)\,(\lambda x e\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e))\, e\, \phi)\\
&\quad\twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\, (\lambda e.\, (\lambda x e\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e))\, x\, e\, \phi)\\
&\quad\twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\, (\lambda e.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e))
\end{aligned}
$$

$[\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))$

$$
\begin{aligned}
&= [\![\text{who}]\!]\,(\lambda s.\, s\,(\lambda xe\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e)))\\
&= (\lambda rnxe\phi.\, n\, x\, e\,(\lambda e.\, r\,(\lambda\psi.\, \psi\, x)\, e\, \phi))\\
&\qquad (\lambda s.\, s\,(\lambda xe\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e)))\\
&\twoheadrightarrow_\beta \lambda nxe\phi.\, n\, x\, e\,(\lambda e.\\
&\qquad (\lambda s.\, s\,(\lambda xe\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e)))\,(\lambda\psi.\, \psi\, x)\, e\, \phi)\\
&\twoheadrightarrow_\beta \lambda nxe\phi.\, n\, x\, e\,(\lambda e.\, (\lambda\psi.\, \psi\, x)\,(\lambda xe\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e))\, e\, \phi)\\
&\twoheadrightarrow_\beta \lambda nxe\phi.\, n\, x\, e\,(\lambda e.\, (\lambda xe\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e))\, x\, e\, \phi)\\
&\twoheadrightarrow_\beta \lambda nxe\phi.\, n\, x\, e\,(\lambda e.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e))
\end{aligned}
$$

$[\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]$

$$[\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))$$

$$= \quad [\![\text{who}]\!]\,(\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e)))$$

$$= \quad (\lambda r n x e \phi.\, n\, x\, e\,(\lambda e.\, r\,(\lambda \psi.\, \psi\, x)\, e\, \phi)) \\ (\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e)))$$

$$\twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e. \\ (\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e)))\,(\lambda \psi.\, \psi\, x)\, e\, \phi)$$

$$\twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, (\lambda \psi.\, \psi\, x)\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e))\, e\, \phi)$$

$$\twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, (\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e))\, x\, e\, \phi)$$

$$\twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e))$$

$$[\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]$$

$$= \quad (\lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e)))\,[\![\text{farmer}]\!]$$

$$[\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))$$

$$\begin{aligned}
&= [\![\text{who}]\!]\,(\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e)))\\
&= (\lambda r n x e \phi.\, n\, x\, e\,(\lambda e.\, r\,(\lambda \psi.\, \psi\, x)\, e\, \phi))\\
&\qquad (\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e)))\\
&\twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\\
&\quad (\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e)))\,(\lambda \psi.\, \psi\, x)\, e\, \phi)\\
&\twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, (\lambda \psi.\, \psi\, x)\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e))\, e\, \phi)\\
&\twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, (\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e))\, x\, e\, \phi)\\
&\twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e))
\end{aligned}$$

$$[\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]$$

$$\begin{aligned}
&= (\lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e)))\,[\![\text{farmer}]\!]\\
&\twoheadrightarrow_\beta \lambda x e \phi.\, [\![\text{farmer}]\!]\, x\, e\,(\lambda e.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e))
\end{aligned}$$

$$[\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))$$

$$\begin{aligned}
&= [\![\text{who}]\!]\,(\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e)))\\
&= (\lambda r n x e \phi.\, n\, x\, e\,(\lambda e.\, r\,(\lambda \psi.\, \psi\, x)\, e\, \phi))\\
&\qquad (\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e)))\\
&\twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\\
&\quad (\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e)))\,(\lambda \psi.\, \psi\, x)\, e\, \phi)\\
&\twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, (\lambda \psi.\, \psi\, x)\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e))\, e\, \phi)\\
&\twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, (\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e))\, x\, e\, \phi)\\
&\twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e))
\end{aligned}$$

$$[\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]$$

$$\begin{aligned}
&= (\lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e)))\,[\![\text{farmer}]\!]\\
&\twoheadrightarrow_\beta \lambda x e \phi.\, [\![\text{farmer}]\!]\, x\, e\,(\lambda e.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e))\\
&= \lambda x e \phi.\, (\lambda x e \phi.\, \mathbf{farmer}\, x \wedge \phi\, e)\, x\, e\,(\lambda e.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e))
\end{aligned}$$

$[\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))$

$$
\begin{aligned}
&= \; [\![\text{who}]\!]\,(\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e)))\\
&= \; (\lambda r n x e \phi.\, n\, x\, e\,(\lambda e.\, r\,(\lambda \psi.\, \psi\, x)\, e\, \phi))\\
&\qquad (\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e)))\\
&\twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\\
&\quad (\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e)))\,(\lambda \psi.\, \psi\, x)\, e\, \phi)\\
&\twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, (\lambda \psi.\, \psi\, x)\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e))\, e\, \phi)\\
&\twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, (\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e))\, x\, e\, \phi)\\
&\twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e))
\end{aligned}
$$

$[\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]$

$$
\begin{aligned}
&= \; (\lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e)))\,[\![\text{farmer}]\!]\\
&\twoheadrightarrow_\beta \lambda x e \phi.\, [\![\text{farmer}]\!]\, x\, e\,(\lambda e.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e))\\
&= \; \lambda x e \phi.\, (\lambda x e \phi.\, \mathbf{farmer}\, x \wedge \phi\, e)\, x\, e\,(\lambda e.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e))\\
&\twoheadrightarrow_\beta \lambda x e \phi.\, \mathbf{farmer}\, x \wedge (\lambda e.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e))\, e
\end{aligned}
$$

$$[\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))$$

$$\begin{aligned}
&= [\![\text{who}]\!]\,(\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \textbf{donkey}\, y \wedge \textbf{own}\, x\, y \wedge \phi\,(y{::}e)))\\
&= (\lambda r n x e \phi.\, n\, x\, e\,(\lambda e.\, r\,(\lambda \psi.\, \psi\, x)\, e\, \phi))\\
&\qquad (\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \textbf{donkey}\, y \wedge \textbf{own}\, x\, y \wedge \phi\,(y{::}e)))\\
&\twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\\
&\qquad (\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \textbf{donkey}\, y \wedge \textbf{own}\, x\, y \wedge \phi\,(y{::}e)))\,(\lambda \psi.\, \psi\, x)\, e\, \phi)\\
&\twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, (\lambda \psi.\, \psi\, x)\,(\lambda x e \phi.\, \exists y.\, \textbf{donkey}\, y \wedge \textbf{own}\, x\, y \wedge \phi\,(y{::}e))\, e\, \phi)\\
&\twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, (\lambda x e \phi.\, \exists y.\, \textbf{donkey}\, y \wedge \textbf{own}\, x\, y \wedge \phi\,(y{::}e))\, x\, e\, \phi)\\
&\twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, \exists y.\, \textbf{donkey}\, y \wedge \textbf{own}\, x\, y \wedge \phi\,(y{::}e))
\end{aligned}$$

$$[\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]$$

$$\begin{aligned}
&= (\lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, \exists y.\, \textbf{donkey}\, y \wedge \textbf{own}\, x\, y \wedge \phi\,(y{::}e)))\,[\![\text{farmer}]\!]\\
&\twoheadrightarrow_\beta \lambda x e \phi.\, [\![\text{farmer}]\!]\, x\, e\,(\lambda e.\, \exists y.\, \textbf{donkey}\, y \wedge \textbf{own}\, x\, y \wedge \phi\,(y{::}e))\\
&= \lambda x e \phi.\, (\lambda x e \phi.\, \textbf{farmer}\, x \wedge \phi\, e)\, x\, e\,(\lambda e.\, \exists y.\, \textbf{donkey}\, y \wedge \textbf{own}\, x\, y \wedge \phi\,(y{::}e))\\
&\twoheadrightarrow_\beta \lambda x e \phi.\, \textbf{farmer}\, x \wedge (\lambda e.\, \exists y.\, \textbf{donkey}\, y \wedge \textbf{own}\, x\, y \wedge \phi\,(y{::}e))\, e\\
&\twoheadrightarrow_\beta \lambda x e \phi.\, \textbf{farmer}\, x \wedge (\exists y.\, \textbf{donkey}\, y \wedge \textbf{own}\, x\, y \wedge \phi\,(y{::}e))
\end{aligned}$$

$[\![\text{every}]\!]\ ([\![\text{who}]\!]\ ([\![\text{owns}]\!]\ ([\![\text{a}]\!]\ [\![\text{donkey}]\!]))\ [\![\text{farmer}]\!])$

$$[\![\text{every}]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!])$$

$$= \quad [\![\text{every}]\!]\,(\lambda x e \phi.\, \mathbf{farmer}\, x \wedge (\exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(y{::}e)))$$

$$
\begin{aligned}
&[\![\text{every}]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]) \\
&\quad = \quad [\![\text{every}]\!]\,(\lambda x e \phi.\,\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge \phi\,(y{::}e))) \\
&\quad = \quad (\lambda n \psi e \phi.\,(\forall x.\,\neg(n\,x\,e\,(\lambda e.\,\neg(\psi\,x\,(x{::}e)\,(\lambda e.\,\top))))) \wedge \phi\,e) \\
&\qquad\qquad (\lambda x e \phi.\,\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge \phi\,(y{::}e)))
\end{aligned}
$$

$$[\![\text{every}]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!])$$

$$= \quad [\![\text{every}]\!]\,(\lambda xe\phi.\,\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge \phi\,(y{::}e)))$$

$$= \quad (\lambda n\psi e\phi.\,(\forall x.\,\neg(n\,x\,e\,(\lambda e.\,\neg(\psi\,x\,(x{::}e)\,(\lambda e.\,\top))))) \wedge \phi\,e)$$
$$(\lambda xe\phi.\,\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge \phi\,(y{::}e)))$$

$$\twoheadrightarrow_\beta \lambda\psi e\phi.\,(\forall x.\,\neg((\lambda xe\phi.\,\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge \phi\,(y{::}e)))$$
$$x\,e\,(\lambda e.\,\neg(\psi\,x\,(x{::}e)\,(\lambda e.\,\top))))) \wedge \phi\,e$$

$[\![\text{every}]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!])$

$$\begin{aligned}
&= [\![\text{every}]\!]\,(\lambda xe\phi.\,\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge \phi\,(y{::}e)))\\
&= (\lambda n\psi e\phi.\,(\forall x.\,\neg(n\,x\,e\,(\lambda e.\,\neg(\psi\,x\,(x{::}e)\,(\lambda e.\,\top))))) \wedge \phi\,e)\\
&\qquad (\lambda xe\phi.\,\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge \phi\,(y{::}e)))\\
&\twoheadrightarrow_\beta \lambda\psi e\phi.\,(\forall x.\,\neg((\lambda xe\phi.\,\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge \phi\,(y{::}e)))\\
&\qquad x\,e\,(\lambda e.\,\neg(\psi\,x\,(x{::}e)\,(\lambda e.\,\top))))) \wedge \phi\,e\\
&\twoheadrightarrow_\beta \lambda\psi e\phi.\,(\forall x.\,\neg(\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge\\
&\qquad (\lambda e.\,\neg(\psi\,x\,(x{::}e)\,(\lambda e.\,\top)))\,(y{::}e)))) \wedge \phi\,e
\end{aligned}$$

$$[\![\text{every}]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!])$$

$$\begin{aligned}
&= [\![\text{every}]\!]\,(\lambda x e \phi.\,\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge \phi\,(y{::}e)))\\
&= (\lambda n \psi e \phi.\,(\forall x.\,\neg(n\,x\,e\,(\lambda e.\,\neg(\psi\,x\,(x{::}e)\,(\lambda e.\,\top))))) \wedge \phi\,e)\\
&\qquad (\lambda x e \phi.\,\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge \phi\,(y{::}e)))\\
&\twoheadrightarrow_\beta \lambda \psi e \phi.\,(\forall x.\,\neg((\lambda x e \phi.\,\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge \phi\,(y{::}e)))\\
&\qquad x\,e\,(\lambda e.\,\neg(\psi\,x\,(x{::}e)\,(\lambda e.\,\top))))) \wedge \phi\,e\\
&\twoheadrightarrow_\beta \lambda \psi e \phi.\,(\forall x.\,\neg(\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge\\
&\qquad (\lambda e.\,\neg(\psi\,x\,(x{::}e)\,(\lambda e.\,\top)))\,(y{::}e)))) \wedge \phi\,e\\
&\twoheadrightarrow_\beta \lambda \psi e \phi.\,(\forall x.\,\neg(\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge\\
&\qquad \neg(\psi\,x\,(x{::}y{::}e)\,(\lambda e.\,\top))))) \wedge \phi\,e
\end{aligned}$$

$[\![\text{beats}]\!]\ [\![\text{it}]\!]$

$$\llbracket \text{beats} \rrbracket \, \llbracket \text{it} \rrbracket$$
$$= \quad (\lambda o s.\, s\, (\lambda x.\, o\, (\lambda y e \phi.\, \mathbf{beat}\, x\, y \;\wedge\; \phi\, e)))\, \llbracket \text{it} \rrbracket$$

$$
\begin{aligned}
&[\![\text{beats}]\!]\,[\![\text{it}]\!] \\
&\quad= \quad (\lambda os.\, s\,(\lambda x.\, o\,(\lambda ye\phi.\, \mathbf{beat}\, x\, y \;\wedge\; \phi\, e)))\,[\![\text{it}]\!] \\
&\quad\twoheadrightarrow_\beta\ \lambda s.\, s\,(\lambda x.\, [\![\text{it}]\!]\,(\lambda ye\phi.\, \mathbf{beat}\, x\, y \;\wedge\; \phi\, e))
\end{aligned}
$$

$$[\![\text{beats}]\!]\,[\![\text{it}]\!]$$

$$= \quad (\lambda os.\, s\,(\lambda x.\, o\,(\lambda ye\phi.\, \mathbf{beat}\, x\, y\, \wedge\, \phi\, e)))\,[\![\text{it}]\!]$$

$$\twoheadrightarrow_\beta\ \lambda s.\, s\,(\lambda x.\, [\![\text{it}]\!]\,(\lambda ye\phi.\, \mathbf{beat}\, x\, y\, \wedge\, \phi\, e))$$

$$= \quad \lambda s.\, s\,(\lambda x.\, (\lambda \psi e\phi.\, \psi\,(\mathtt{sel}_{it}\, e)\, e\, \phi)\,(\lambda ye\phi.\, \mathbf{beat}\, x\, y\, \wedge\, \phi\, e))$$

$$
\begin{aligned}
&[\![\text{beats}]\!]\,[\![\text{it}]\!] \\
&\quad = \quad (\lambda os.\, s\,(\lambda x.\, o\,(\lambda ye\phi.\, \mathbf{beat}\, x\, y \,\wedge\, \phi\, e)))\,[\![\text{it}]\!] \\
&\quad \twoheadrightarrow_\beta \lambda s.\, s\,(\lambda x.\, [\![\text{it}]\!]\,(\lambda ye\phi.\, \mathbf{beat}\, x\, y \,\wedge\, \phi\, e)) \\
&\quad = \quad \lambda s.\, s\,(\lambda x.\, (\lambda \psi e\phi.\, \psi\,(\texttt{sel}_{it}\, e)\, e\, \phi)\,(\lambda ye\phi.\, \mathbf{beat}\, x\, y \,\wedge\, \phi\, e)) \\
&\quad \twoheadrightarrow_\beta \lambda s.\, s\,(\lambda xe\phi.\, (\lambda ye\phi.\, \mathbf{beat}\, x\, y \,\wedge\, \phi\, e)\,(\texttt{sel}_{it}\, e)\, e\, \phi)
\end{aligned}
$$

$[\![\text{beats}]\!]\,[\![\text{it}]\!]$

$$
\begin{aligned}
&= \quad (\lambda os.\, s\,(\lambda x.\, o\,(\lambda ye\phi.\, \mathbf{beat}\, x\, y \,\wedge\, \phi\, e)))\,[\![\text{it}]\!] \\
&\twoheadrightarrow_\beta \lambda s.\, s\,(\lambda x.\, [\![\text{it}]\!]\,(\lambda ye\phi.\, \mathbf{beat}\, x\, y \,\wedge\, \phi\, e)) \\
&= \quad \lambda s.\, s\,(\lambda x.\, (\lambda \psi e\phi.\, \psi\,(\mathtt{sel}_{it}\, e)\, e\, \phi)\,(\lambda ye\phi.\, \mathbf{beat}\, x\, y \,\wedge\, \phi\, e)) \\
&\twoheadrightarrow_\beta \lambda s.\, s\,(\lambda xe\phi.\, (\lambda ye\phi.\, \mathbf{beat}\, x\, y \,\wedge\, \phi\, e)\,(\mathtt{sel}_{it}\, e)\, e\, \phi) \\
&\twoheadrightarrow_\beta \lambda s.\, s\,(\lambda xe\phi.\, \mathbf{beat}\, x\,(\mathtt{sel}_{it}\, e) \,\wedge\, \phi\, e)
\end{aligned}
$$

$[\![\text{beats}]\!]\ [\![\text{it}]\!]\ ([\![\text{every}]\!]\ ([\![\text{who}]\!]\ ([\![\text{owns}]\!]\ ([\![\text{a}]\!]\ [\![\text{donkey}]\!]))\ [\![\text{farmer}]\!]))$

$$[\![\text{beats}]\!]\,[\![\text{it}]\!]\,([\![\text{every}]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))$$

$$= \quad (\lambda s.\, s\,(\lambda xe\phi.\, \mathbf{beat}\, x\,(\mathtt{sel}_{it}\, e) \wedge \phi\, e))$$

$$([\![\text{every}]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))$$

$$
\begin{aligned}
&[\![\text{beats}]\!]\,[\![\text{it}]\!]\,([\![\text{every}]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))\\
&\quad = \quad (\lambda s.\, s\,(\lambda x e \phi.\, \mathbf{beat}\, x\,(\texttt{sel}_{it}\, e) \wedge \phi\, e))\\
&\qquad\qquad ([\![\text{every}]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))\\
&\quad \twoheadrightarrow_\beta [\![\text{every}]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!])\\
&\qquad\qquad (\lambda x e \phi.\, \mathbf{beat}\, x\,(\texttt{sel}_{it}\, e) \wedge \phi\, e)
\end{aligned}
$$

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\begin{aligned}
&[\![\text{beats}]\!]\,[\![\text{it}]\!]\,([\![\text{every}]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))\\
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&\qquad\qquad (\lambda x e \phi.\, \mathbf{beat}\, x\,(\mathtt{sel}_{it}\, e) \wedge \phi\, e)\\
&\quad = \quad (\lambda \psi e \phi.\, (\forall x.\, \neg(\mathbf{farmer}\, x \wedge (\exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge\\
&\qquad\qquad \neg(\psi\, x\,(x{::}y{::}e)\,(\lambda e.\, \top))))) \wedge \phi\, e)\,(\lambda x e \phi.\, \mathbf{beat}\, x\,(\mathtt{sel}_{it}\, e) \wedge \phi\, e)
\end{aligned}
$$

$$[\![\text{beats}]\!]\,[\![\text{it}]\!]\,([\![\text{every}]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))$$

$$= \quad (\lambda s.\, s\,(\lambda xe\phi.\, \mathbf{beat}\, x\,(\mathtt{sel}_{it}\, e) \wedge \phi\, e))$$
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$$[\![\text{beats}]\!]\,[\![\text{it}]\!]\,([\![\text{every}]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))$$

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$$
\begin{aligned}
&[\![\text{beats}]\!]\,[\![\text{it}]\!]\,([\![\text{every}]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))\\
&\quad = \;(\lambda s.\, s\,(\lambda x e \phi.\, \mathbf{beat}\,x\,(\mathtt{sel}_{it}\,e) \wedge \phi\,e))\\
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&\quad \twoheadrightarrow_\beta \lambda e \phi.\, (\forall x.\, \neg(\mathbf{farmer}\,x \wedge (\exists y.\, \mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge\\
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\end{aligned}
$$

$$[\![\text{beats}]\!]\, [\![\text{it}]\!]\, ([\![\text{every}]\!]\, ([\![\text{who}]\!]\, ([\![\text{owns}]\!]\, ([\![\text{a}]\!]\, [\![\text{donkey}]\!]))\, [\![\text{farmer}]\!]))$$

$$\begin{aligned}
&= (\lambda s.\, s\, (\lambda x e \phi.\, \mathbf{beat}\, x\, (\mathtt{sel}_{it}\, e) \wedge \phi\, e)) \\
&\qquad ([\![\text{every}]\!]\, ([\![\text{who}]\!]\, ([\![\text{owns}]\!]\, ([\![\text{a}]\!]\, [\![\text{donkey}]\!]))\, [\![\text{farmer}]\!])) \\
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&\qquad (\lambda x e \phi.\, \mathbf{beat}\, x\, (\mathtt{sel}_{it}\, e) \wedge \phi\, e) \\
&= (\lambda \psi e \phi.\, (\forall x.\, \neg(\mathbf{farmer}\, x \wedge (\exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \\
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&[\![\text{beats}]\!]\,[\![\text{it}]\!]\,([\![\text{every}]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))\\
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&\quad \equiv \quad \lambda e\phi.\, (\forall x.\, \mathbf{farmer}\,x \supset (\forall y.\, (\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y) \supset \mathbf{beat}\,x\,y)) \wedge \phi\,e
\end{aligned}
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[\![\text{every}]\!] &= \lambda n \psi e \phi.\, (\forall x.\, \neg(n\, x\, e\, (\lambda e.\, \neg(\psi\, x\, (x{::}e)\, (\lambda e.\, \top))))) \wedge \phi\, e \\
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...which might seem a little bit involved.

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A Dynamic Logic

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We share with DRT the two following assumptions:

- discourse composition is mainly conjunctive (roughly speaking, a discourse consists in the conjunction of its sentences);
- the main form of quantification is existential (it introduces referential markers).

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- discourse composition is mainly conjunctive (roughly speaking, a discourse consists in the conjunction of its sentences);
- the main form of quantification is existential (it introduces referential markers).

Consequently, our logic will be based on conjunction and existential quantification (defined as primitives). The other connectives will be obtained using negation (a third primitive) and de Morgan's laws.

Formal Framework

We consider a simply-typed λ-calculus, the terms of which are built upon asignature including the following constants:

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FIRST-ORDER LOGIC

$$\top \; : \; o \qquad (truth)$$

$$\neg \; : \; o \rightarrow o \qquad (negation)$$

$$\wedge \; : \; o \rightarrow o \rightarrow o \qquad (conjunction)$$

$$\exists \; : \; (\iota \rightarrow o) \rightarrow o \qquad (existential\ quantification)$$

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DYNAMIC PRIMITIVES

$$\begin{array}{rcll} :: & : & \iota \to \gamma \to \gamma & (\textit{context updating}) \\ \mathsf{sel} & : & \gamma \to \iota & (\textit{choice operator}) \end{array}$$

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Existential quantification introduces "reference markers". It is therefore responsible for context updating:

$$\Sigma x.\, P\, x \triangleq \lambda e\phi.\, \exists x.\, P\, x\, (x{::}e)\, \phi$$

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Negation

We do not want the continuation of the discourse to fall into the scope of the negation. Consequently, negation must be defined as follows:

$$\sim A \triangleq \lambda e\phi.\, \neg\, (A\, e\, (\lambda e.\, \top)) \wedge \phi\, e$$

Implication and Universal Quantification

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These are defined using de Morgan's laws:

$$
\begin{aligned}
A \sqsupset B &\triangleq \sim(A \sqcap \sim B) \\
\Pi x.\, P\, x &\triangleq \sim\Sigma x.\, \sim(P\, x)
\end{aligned}
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Embedding of first-order logic into dynamic logic

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Embedding of first-order logic into dynamic logic

$$\begin{aligned} \overline{R\, t_1\, \ldots\, t_n} &= \lambda e\phi.\, R\, t_1\, \ldots\, t_n \wedge \phi\, e \\ \overline{\neg A} &= {\sim}\overline{A} \\ \overline{A \wedge B} &= \overline{A} \sqcap \overline{B} \\ \overline{\exists x.\, A} &= \Sigma x.\, \overline{A} \end{aligned}$$

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\end{aligned}$$

This embedding is such that, for every term e of type γ:

$$A \equiv \overline{A}\, e\, (\lambda e.\, \top)$$

Donkey Sentence Revisited

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Montague-like semantic interpretation:

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\begin{aligned}
[\![\text{farmer}]\!] &= \mathbf{farmer} \\
[\![\text{donkey}]\!] &= \mathbf{donkey} \\
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[\![\text{beats}]\!] &= \lambda OS.\, S\,(\lambda x.\, O\,(\lambda y.\, \mathbf{beat}\, x\, y)) \\
[\![\text{who}]\!] &= \lambda RQx.\, Q\, x \wedge R\,(\lambda P.\, P\, x) \\
[\![\text{a}]\!] &= \lambda PQ.\, \exists x.\, P\, x \wedge Q\, x \\
[\![\text{every}]\!] &= \lambda PQ.\, \forall x.\, P\, x \supset Q\, x \\
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[\![\text{a}]\!] &= \lambda PQ.\, \Sigma x.\, P\, x \sqcap Q\, x \\
[\![\text{every}]\!] &= \lambda PQ.\, \Pi x.\, P\, x \sqsupset Q\, x \\
[\![\text{it}]\!] &= \lambda Pe\phi.\, P\,(\texttt{sel}\, e)\, e\, \phi
\end{aligned}
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With the dynamic interpretation we have that:

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that is, assuming that sel is a "perfect" anaphora resolution operator:

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Define type "dynamization" as follows:

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\mathsf{D}\iota &= \iota \\
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\mathsf{D}t &= \lambda x_1 \ldots x_n.\, t\,(\mathsf{R}_{\mathsf{nil}} x_1) \ldots (\mathsf{R}_{\mathsf{nil}} x_n) && \text{at type } \alpha_1 \to \ldots \alpha_n \to \iota \\
\mathsf{D}t &= \lambda x_1 \ldots x_n e \phi.\, t\,(\mathsf{R}_e x_1) \ldots (\mathsf{R}_e x_n) \wedge (\phi\, e) && \text{at type } \alpha_1 \to \ldots \alpha_n \to o \\
\mathsf{R}_e t &= \lambda x_1 \ldots x_n.\, t\,(\mathsf{D} x_1) \ldots (\mathsf{D} x_n) && \text{at type } \mathsf{D}(\alpha_1 \to \ldots \alpha_n \to \iota) \\
\mathsf{R}_e t &= \lambda x_1 \ldots x_n.\, t\,(\mathsf{D} x_1) \ldots (\mathsf{D} x_n)\, e\, (\lambda e.\, \top) && \text{at type } \mathsf{D}(\alpha_1 \to \ldots \alpha_n \to o)
\end{aligned}
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Finally, define λ-term translation as follows:

$$\begin{array}{rcll} \overline{x} & = & x & \\ \overline{\wedge} & = & \sqcap & \\ \overline{\exists} & = & \Sigma & \\ \overline{\neg} & = & \sim & \\ \overline{k} & = & \mathsf{D}k & \text{for the other constants} \\ \overline{\lambda x.\, t} & = & \lambda x.\, \overline{t} & \\ \overline{t\, u} & = & \overline{t}\, \overline{u} & \end{array}$$

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Then, for every closed term t of type o, and every context e, we have that:

$$\overline{t}\, e\, (\lambda e.\, \top) \equiv t$$

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and the semantics of Groenendijk & Stokhof's DPL would be rephrased as follows:

$$\begin{array}{rcll}
A_{\mathrm{d}} & \triangleq & \lambda gh.\, h{=}g \wedge A & \text{(atomic proposition)} \\
(\neg P)_{\mathrm{d}} & \triangleq & \lambda gh.\, h{=}g \wedge \neg(\exists k.\, P_{\mathrm{d}}\, h\, k) & \text{(negation)} \\
(P \wedge Q)_{\mathrm{d}} & \triangleq & \lambda gh.\, \exists k.\, P_{\mathrm{d}}\, g\, k \wedge Q_{\mathrm{d}}\, k\, h & \text{(conjunction)} \\
(\exists x.\, P)_{\mathrm{d}} & \triangleq & \lambda gh.\, \exists k.\, k[x]g \wedge P_{\mathrm{d}}\, k\, h & \text{(existential)}
\end{array}$$

There exists a canonical embedding $\lceil\cdot\rceil$ from $\gamma \to \gamma \to o$ into $\gamma \to (\gamma \to o) \to o$:

$$\lceil R \rceil \triangleq \lambda e\phi.\, \exists e'.\, \phi\, e' \wedge R\, e\, e'$$

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As for the existential quantifier:

$$\begin{aligned} \lceil (\exists x.\, P)_{\mathrm{d}} \rceil &= \lambda e \phi.\, \exists e'.\, \phi\, e' \wedge \exists k.\, k[x]e \wedge \lceil P_{\mathrm{d}} \rceil\, k\, e' \\ \Sigma x.\, \lceil P_{\mathrm{d}} \rceil &= \lambda e \phi.\, \exists e'.\, \phi\, e' \wedge \exists x.\, \lceil P_{\mathrm{d}} \rceil\, (x :: e)\, e' \end{aligned}$$

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- Deduction is replaced by computation.